Algebraic Number Theory

(PARI-GP version 2.9.0)

Binary Quadratic Forms

create $ax^2+bxy+cy^2$ (distance d)	`Qfb(a,b,c,{d})`
reduce x ($s=\sqrt{D}$, $l=\lfloor s\rfloor$)	`qfbred(x,{flag},{D},{l},{s})`
return $[y,g]$, $g\in \mathrm{SL}_2(\mathbf{Z})$, $y=g\cdot x$ reduced	`qfbredsl2(x)`
composition of forms	`x*y` or `qfbnucomp(x,y,l)`
n-th power of form	`x^n` or `qfbnupow(x,n)`
composition without reduction	`qfbcompraw(x,y)`
n-th power without reduction	`qfbpowraw(x,n)`
prime form of disc. x above prime p	`qfbprimeform(x,p)`
class number of disc. x	`qfbclassno(x)`
Hurwitz class number of disc. x	`qfbhclassno(x)`
Solve $Q(x,y)=p$ in integers, p prime	`qfbsolve(Q,p)`

Quadratic Fields

quadratic number $\omega=\sqrt{x}$ or $(1+\sqrt{x})/2$	`quadgen(x)`
minimal polynomial of ω	`quadpoly(x)`
discriminant of $\mathbf{Q}(\sqrt{D})$	`quaddisc(x)`
regulator of real quadratic field	`quadregulator(x)`
fundamental unit in real $\mathbf{Q}(x)$	`quadunit(x)`
class group of $\mathbf{Q}(\sqrt{D})$	`quadclassunit(D,{flag},{t})`
Hilbert class field of $\mathbf{Q}(\sqrt{D})$	`quadhilbert(D,{flag})`
... using specific class invariant ($D<0$)	`polclass(D,{inv})`
ray class field modulo f of $\mathbf{Q}(\sqrt{D})$	`quadray(D,f,{flag})`

General Number Fields: Initializations

The number field $K=\mathbf{Q}[X]/(f)$ is given by irreducible $f\in\mathbf{Q}[X]$. A *nf* computes a maximal order and allows operations on elements and ideals. A *bnf* adds class group and units. A *bnr* is attached to ray class groups and class field theory. A *rnf* is attached to relative extensions L/K.

init number field structure *nf*	`nfinit(f,{flag})`
known integer basis B	`nfinit([f,B])`
order maximal at $vp=[p_1,\dots,p_k]$	`nfinit([f,vp])`
order maximal at all $p\le P$	`nfinit([f,P])`
certify maximal order	`nfcertify(nf)`

nf members:

a monic $F\in\mathbf{Z}[X]$ defining K	`nf.pol`
number of real/complex places	`nf.r1/r2/sign`
discriminant of *nf*	`nf.disc`
T_2 matrix	`nf.t2`
complex roots of F	`nf.roots`
integral basis of $\mathbf{Z}_K$ as powers of θ	`nf.zk`
different/codifferent	`nf.diff`, `nf.codiff`
index $[\mathbf{Z}_K:\mathbf{Z}[X]/(F)]$	`nf.index`
recompute *nf* using current precision	`nfnewprec(nf)`
init relative *rnf* $L=K[Y]/(g)$	`rnfinit(nf,g)`
init *bnf* structure	`bnfinit(f,{flag})`

bnf members: same as *nf*, plus

underlying *nf*	`bnf.nf`
classgroup	`bnf.clgp`
regulator	`bnf.reg`
fundamental/torsion units	`bnf.fu`, `bnf.tu`
compress a *bnf* for storage	`bnfcompress(bnf)`
recover a *bnf* from compressed *bnfz*	`bnfinit(bnfz)`
add S-class group and units, yield *bnf*S	`bnfsunit(bnf,S)`
init class field structure *bnr*	`bnrinit(bnf,m,{flag})`

bnr members: same as *bnf*, plus

underlying *bnf*	`bnr.bnf`
big ideal structure	`bnr.bid`
modulus	`bnr.mod`
structure of $(\mathbf{Z}_K/m)^*$	`bnr.zkst`

Basic Number Field Arithmetic (nf)

Elements are `t_INT`, `t_FRAC`, `t_POL`, `t_POLMOD`, or `t_COL` (on integral basis `nf.zk`). Basic operations (prefix `nfelt`): `(nfelt)add`, `mul`, `pow`, `div`, `diveuc`, `mod`, `divrem`, `val`, `trace`, `norm`

express x on integer basis	`nfalgtobasis(nf,x)`
express element x as a polmod	`nfbasistoalg(nf,x)`
complex embeddings of `t_POLMOD` x	`conjvec(x)`
reverse polmod $a=A(X)$ mod $T(X)$	`modreverse(a)`
integral basis of field def. by $f=0$	`nfbasis(f)`
field discriminant of field $f=0$	`nfdisc(f)`
smallest poly defining $f=0$ (slow)	`polredabs(f,{flag})`
small poly defining $f=0$ (fast)	`polredbest(f,{flag})`
random Tschirnhausen transform of f	`poltschirnhaus(f)`
$\mathbf{Q}[x]/(f)\subset\mathbf{Q}[x]/(g)$? Isomorphic?	`nfisincl(f,g)`, `nfisisom`
compositum of $\mathbf{Q}[X]/(f)$, $\mathbf{Q}[X]/(g)$	`polcompositum(f,g,{flag})`
compositum of $K[X]/(f)$, $K[X]/(g)$	`nfcompositum(nf,f,g,{flag})`
splitting field of K (degree divides d)	`nfsplitting(nf,{d})`
subfields (of degree d) of *nf*	`nfsubfields(nf,{d})`
d-th degree subfield of $\mathbf{Q}(\zeta_n)$	`polsubcyclo(n,d,{v})`
roots of unity in *nf*	`nfrootsof1(nf)`
roots of g belonging to *nf*	`nfroots({nf},g)`
factor g in *nf*	`nffactor(nf,g)`
factor g mod prime *pr* in *nf*	`nffactormod(nf,g,pr)`
conjugates of a root θ of *nf*	`nfgaloisconj(nf,{flag})`
apply Galois automorphism s to x	`nfgaloisapply(nf,s,x)`
quadratic Hilbert symbol (at p)	`nfhilbert(nf,a,b,{p})`

Linear and algebraic relations

poly of degree $\le k$ with root $x\in\mathbf{C}$	`algdep(x,k)`
alg. dep. with pol. coeffs for series s	`seralgdep(s,x,y)`
small linear rel. on coords of vector x	`lindep(x)`

Dedekind Zeta Function ζ_K, Hecke L series

$R=[c,w,h]$ in initialization means we restrict $s\in\mathbf{C}$ to domain $|\Re(s)-c|<w$, $|\Im(s)|<h$; $R=[w,h]$ encodes $[1/2,w,h]$ and $[h]$ encodes $R=[1/2,0,h]$ (critical line up to height h).

ζ_K as Dirichlet series, $N(I)<b$	`dirzetak(nf,b)`
init $\zeta_K^{(k)}(s)$ for $k\le n$	`L = lfuninit(bnf,R,{n=0})`
compute $\zeta_K(s)$ (n-th derivative)	`lfun(L,s,{n=0})`
compute $\Lambda_K(s)$ (n-th derivative)	`lfunlambda(L,s,{n=0})`
init $L_K^{(k)}(s,\chi)$ for $k\le n$	`L = lfuninit([bnr,chi],R,{n=0})`
compute $L_K(s,\chi)$ (n-th derivative)	`lfun(L,s,{n})`
Artin root number of K	`bnrrootnumber(bnr,chi,{flag})`
$L(1,\chi)$, for all χ trivial on H	`bnrL1(bnr,{H},{flag})`

Class Groups & Units (bnf, bnr)

Class field theory data $a_1,\{a_2\}$ is usually *bnr* (ray class field), *bnr*, H (congruence subgroup) or *bnr*, χ (character on `bnr.clgp`). Any of these define a unique abelian extension of K.

remove GRH assumption from *bnf*	`bnfcertify(bnf)`
expo. of ideal x on class gp	`bnfisprincipal(bnf,x,{flag})`
expo. of ideal x on ray class gp	`bnrisprincipal(bnr,x,{flag})`
expo. of x on fund. units	`bnfisunit(bnf,x)`
as above for S-units	`bnfissunit(bnfs,x)`
signs of real embeddings of `bnf.fu`	`bnfsignunit(bnf)`
narrow class group	`bnfnarrow(bnf)`

Class Field Theory

ray class number for modulus m	`bnrclassno(bnf,m)`
discriminant of class field	`bnrdisc(a1,{a2})`
ray class numbers, l list of moduli	`bnrclassnolist(bnf,l)`
discriminants of class fields	`bnrdisclist(bnf,l,{arch},{flag})`
decode output from `bnrdisclist`	`bnfdecodemodule(nf,fa)`
is modulus the conductor?	`bnrisconductor(a1,{a2})`
is class field (bnr,H) Galois over K^G	`bnrisgalois(bnr,G,H)`
action of automorphism on `bnr.gen`	`bnrgaloismatrix(bnr,aut)`
apply `bnrgaloismatrix` M to H	`bnrgaloisapply(bnr,M,H)`
characters on `bnr.clgp` s.t. $\chi(g_i)=e(v_i)$	`bnrchar(bnr,g,{v})`
conductor of character χ	`bnrconductor(bnr,chi)`
conductor of extension	`bnrconductor(a1,{a2},{flag})`
conductor of extension $K[Y]/(g)$	`rnfconductor(bnf,g)`
Artin group of extension $K[Y]/(g)$	`rnfnormgroup(bnr,g)`
subgroups of *bnr*, index $<=b$	`subgrouplist(bnr,b,{flag})`
rel. eq. for class field def'd by *sub*	`rnfkummer(bnr,sub,{d})`
same, using Stark units (real field)	`bnrstark(bnr,sub,{flag})`
is a an n-th power in K_v ?	`nfislocalpower(nf,v,a,n)`
cyclic L/K satisf. local conditions	`nfgrunwaldwang(nf,P,D,pl)`

Logarithmic class group

logarithmic ℓ-class group	`bnflog(bnf,ℓ)`
$[\tilde e(F_v/Q_p),\tilde f(F_v/Q_p)]$	`bnflogef(bnf,pr)`
$\exp\deg_F(A)$	`bnflogdegree(bnf,A,ℓ)`
is ℓ-extension L/K locally cyclotomic	`rnfislocalcyclo(rnf)`

Ideals: elements, primes, or matrix of generators in HNF

is *id* an ideal in *nf* ?	`nfisideal(nf,id)`
is x principal in *bnf* ?	`bnfisprincipal(bnf,x)`
give $[a,b]$, s.t. $a\mathbf{Z}_K+b\mathbf{Z}_K=x$	`idealtwoelt(nf,x,{a})`
put ideal a ($a\mathbf{Z}_K+b\mathbf{Z}_K$) in HNF form	`idealhnf(nf,a,{b})`
norm of ideal x	`idealnorm(nf,x)`
minimum of ideal x (direction v)	`idealmin(nf,x,v)`
LLL-reduce the ideal x (direction v)	`idealred(nf,x,{v})`

Ideal Operations

add ideals x and y	`idealadd(nf,x,y)`
multiply ideals x and y	`idealmul(nf,x,y,{flag})`
intersection of ideals x and y	`idealintersect(nf,x,y,{flag})`
n-th power of ideal x	`idealpow(nf,x,n,{flag})`
inverse of ideal x	`idealinv(nf,x)`
divide ideal x by y	`idealdiv(nf,x,y,{flag})`
Find $(a,b)\in x\times y$, $a+b=1$	`idealaddtoone(nf,x,{y})`
coprime integral A,B such that $x=A/B$	`idealnumden(nf,x)`

Primes and Multiplicative Structure

factor ideal x in $\mathbf{Z}_K$	`idealfactor(nf,x)`
expand ideal factorization in K	`idealfactorback(nf,f,{e})`
expand elt factorisation in K	`nffactorback(nf,f,{e})`
decomposition of prime p in $\mathbf{Z}_K$	`idealprimedec(nf,p)`
valuation of x at prime ideal *pr*	`idealval(nf,x,pr)`
weak approximation theorem in *nf*	`idealchinese(nf,x,y)`
$a\in K$, s.t. $v_{\mathfrak p}(a)=v_{\mathfrak p}(x)$ if $v_{\mathfrak p}(x)\neq 0$	`idealappr(nf,x)`
$a\in K$ such that $(a\cdot x,y)=1$	`idealcoprime(nf,x,y)`
give *bid* =structure of $(\mathbf{Z}_K/id)^*$	`idealstar(nf,id,{flag})`
structure of $(1+\mathfrak p)/(1+\mathfrak p^k)$	`idealprincipalunits(nf,pr,k)`
discrete log of x in $(\mathbf{Z}_K/bid)^*$	`ideallog(nf,x,bid)`

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`idealstar` of all ideals of norm $\leq b$	`ideallist`$(nf, b, \{flag\})$
add Archimedean places	`ideallistarch`$(nf, b, \{ar\}, \{flag\})$
init `modpr` structure	`nfmodprinit`(nf, pr)
project t to $\mathbf{Z}_K/pr$	`nfmodpr`$(nf, t, modpr)$
lift from $\mathbf{Z}_K/pr$	`nfmodprlift`$(nf, t, modpr)$

Galois theory over Q

Galois group of field $\mathbf{Q}[x]/(f)$	`polgalois`(f)
initializes a Galois group structure G	`galoisinit`$(pol, \{den\})$
action of p in nfgaloisconj form	`galoispermtopol`$(G, \{p\})$
identify as abstract group	`galoisidentify`(G)
export a group for GAP/MAGMA	`galoisexport`$(G, \{flag\})$
subgroups of the Galois group G	`galoissubgroups`(G)
is subgroup H normal?	`galoisisnormal`(G, H)
subfields from subgroups	`galoissubfields`$(G, \{flag\}, \{v\})$
fixed field	`galoisfixedfield`$(G, perm, \{flag\}, \{v\})$
Frobenius at maximal ideal P	`idealfrobenius`(nf, G, P)
ramification groups at P	`idealramgroups`(nf, G, P)
is G abelian?	`galoisisabelian`$(G, \{flag\})$
abelian number fields/$\mathbf{Q}$	`galoissubcyclo(N,H,{flag},{v})`
query the `galpol` package	`galoisgetpol(a,b,{s})`

Relative Number Fields (rnf)

Extension L/K is defined by $T \in K[x]$.

absolute equation of L	`rnfequation`$(nf, T, \{flag\})$
is L/K abelian?	`rnfisabelian`(nf, T)
relative `nfalgtobasis`	`rnfalgtobasis`(rnf, x)
relative `nfbasistoalg`	`rnfbasistoalg`(rnf, x)
relative `idealhnf`	`rnfidealhnf`(rnf, x)
relative `idealmul`	`rnfidealmul`(rnf, x, y)
relative `idealtwoelt`	`rnfidealtwoelt`(rnf, x)

Lifts and Push-downs

absolute $\rightarrow$ relative repres. for x	`rnfeltabstorel`(rnf, x)
relative $\rightarrow$ absolute repres. for x	`rnfeltreltoabs`(rnf, x)
lift x to the relative field	`rnfeltup`(rnf, x)
push x down to the base field	`rnfeltdown`(rnf, x)

idem for x ideal: `(rnfideal)reltoabs`, `abstorel`, `up`, `down`

Norms and Trace

relative norm of element $x \in L$	`rnfeltnorm`(rnf, x)
relative trace of element $x \in L$	`rnfelttrace`(rnf, x)
absolute norm of ideal x	`rnfidealnormabs`(rnf, x)
relative norm of ideal x	`rnfidealnormrel`(rnf, x)
solutions of $N_{K/\mathbf{Q}}(y) = x \in \mathbf{Z}$	`bnfisintnorm`(bnf, x)
is $x \in \mathbf{Q}$ a norm from K?	`bnfisnorm`$(bnf, x, \{flag\})$
initialize T for norm eq. solver	`rnfisnorminit`$(K, pol, \{flag\})$
is $a \in K$ a norm from L?	`rnfisnorm`$(T, a, \{flag\})$
initialize t for Thue equation solver	`thueinit`(f)
solve Thue equation $f(x, y) = a$	`thue`$(t, a, \{sol\})$
characteristic poly. of a mod T	`rnfcharpoly`$(nf, T, a, \{v\})$

Factorization

factor ideal x in L	`rnfidealfactor`(rnf, x)
$[S, T]$: $T_{i,j} \mid S_i$; S primes of K above p	`rnfidealprimedec`(rnf, p)

Maximal order $\mathbf{Z}_L$ as a $\mathbf{Z}_K$-module

relative `polredbest`	`rnfpolredbest`(nf, T)
relative Dedekind criterion, prime pr	`rnfdedekind`(nf, T, pr)
discriminant of relative extension	`rnfdisc`(nf, T)
pseudo-basis of $\mathbf{Z}_L$	`rnfpseudobasis`(nf, T)

General $\mathbf{Z}_K$-modules: $M = [\text{matrix}, \text{vec. of ideals}] \subset L$

relative HNF / SNF	`nfhnf`(nf, M), `nfsnf`
multiple of det M	`nfdetint`(nf, M)
HNF of M where $d = nfdetint(M)$	`nfhnfmod`(x, d)
reduced basis for M	`rnflllgram`(nf, T, M)
determinant of pseudo-matrix M	`rnfdet`(nf, M)
Steinitz class of M	`rnfsteinitz`(nf, M)
$\mathbf{Z}_K$-basis of M if $\mathbf{Z}_K$-free, or 0	`rnfhnfbasis`(bnf, M)
n-basis of M, or $(n+1)$-generating set	`rnfbasis`(bnf, M)
is M a free $\mathbf{Z}_K$-module?	`rnfisfree`(bnf, M)

Associative Algebras

A is a general associative algebra given by a mult. table mt (over $\mathbf{Q}$ or $\mathbf{F}_p$); represented by al from `algtableinit`.

create al from mt (over $\mathbf{F}_p$)	`algtableinit`$(mt, \{p = 0\})$
group algebra $\mathbf{Q}[G]$ (or $\mathbf{F}_p[G]$)	`alggroup`$(G, \{p = 0\})$

Properties

is (mt, p) OK for algtableinit?	`algisassociative`$(mt, \{p = 0\})$
multiplication table mt	`algmultable`(al)
multiplication table over center	`algrelmultable`(al)
dimension of A over prime subfield	`algabsdim`(al)
characteristic of A	`algchar`(al)
is A commutative?	`algiscommutative`(al)
is A simple?	`algissimple`(al)
is A semi-simple?	`algissemisimple`(al)
is A ramified? (at place v)	`algisramified`$(al, \{v\})$
is A split? (at place v)	`algissplit`$(al, \{v\})$
center of A	`algcenter`(al)
Jacobson radical of A	`algradical`(al)
radical J and simple factors of A/J	`algdecomposition`(al)
simple factors of semi-simple A	`algsimpledec`(al)

Operations on algebras

create A/I, I two-sided ideal	`algquotient`$(al, I, \{flag = 0\})$
create $A_1 \otimes A_2$	`algtensor`$(al1, al2)$
create subalgebra from basis B	`algsubalg`(al, B)
…from orthogonal central idempotents e	`algcentralproj`(al, e)
prime subalgebra of semi-simple A over $\mathbf{F}_p$	`algprimesubalg`(al)
lattice generated by cols. of M	`alglathnf`(al, M)

Operations on elements

$a + b$, $a - b$, $-a$	`algadd`(al, a, b), `algsub`, `algneg`
$a \times b$, $a \times a$	`algmul`(al, a, a), `algsqr`
a^n, a^{-1}	`algpow`(al, a, n), `alginv`
is x invertible ? (then set $z = x^{-1}$)	`algisinv`$(al, x, \{\&z\})$
find z such that $x \times z = y$	`algdivl`(al, x, y)
find z such that $z \times x = y$	`algdivr`(al, x, y)
does z s.t. $x \times z = y$ exist? (set it)	`algisdivl`$(al, x, y, \{\&z\})$
matrix of $v \mapsto x \cdot v$	`algleftmultable`(al, x)
absolute norm	`algnorm`(al, x)
absolute trace	`algtrace`(al, x)
absolute char. polynomial	`algcharpoly`(al, x)
given $a \in A$ and polynomial T, return $T(a)$	`algpoleval`(al, T, a)
random element in a box	`algrandom`(al, b)

Central Simple Algebras

A is a central simple algebra over a number field K; represented by al from `alginit`; K is given by a nf structure.

create CSA from data	`alginit`$(B, C, \{v\}, \{flag = 0\})$
multiplication table over K	$B = K$, $C = mt$
cyclic algebra $(L/K, \sigma, b)$	$B = rnf$, $C = [sigma, b]$
quaternion algebra $(a, b)_K$	$B = K$, $C = [a, b]$
matrix algebra $M_d(K)$	$B = K$, $C = d$
local Hasse invariants over K	$B = K$, $C = [d, [PR, HF], HI]$

Properties

type of al (mt, CSA)	`algtype`(al)
is al a division algebra? (at place v)	`algisdivision`$(al, \{v\})$
dimension of al over its center	`algdim`(al)
degree of A ($= \sqrt{\dim}$)	`algdegree`(al)
index of A over K (index at v)	`algindex`$(al, \{v\})$
al a cyclic algebra $(L/K, \sigma, b)$; return σ	`algaut`(al)
…return b	`algb`(al)
…return L/K, as an rnf	`algsplittingfield`(al)
split A over an extension of K	`algsplittingdata`(al)
splitting field of A as an rnf over center	`algsplittingfield`(al)
places of K at which A ramifies	`algramifiedplaces`(al)
Hasse invariants at finite places of K	`alghassef`(al)
Hasse invariants at infinite places of K	`alghassei`(al)
Hasse invariant at place v	`alghasse`(al, v)

Operations on elements

reduced norm	`algnorm`(al, x)
reduced trace	`algtrace`(al, x)
reduced char. polynomial	`algcharpoly`(al, x)
express x on integral basis	`algalgtobasis`(al, x)
convert x to algebraic form	`algbasistoalg`(al, x)
map $x \in A$ to $M_d(L)$, L split. field	`algsplittingmatrix`(al, x)

Orders

$\mathbf{Z}$-basis of order $\mathcal{O}_0$	`algbasis`(al)
discriminant of order $\mathcal{O}_0$	`algdisc`(al)
$\mathbf{Z}$-basis of natural order in terms $\mathcal{O}_0$'s basis	`alginvbasis`(al)

Based on an earlier version by Joseph H. Silverman

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